Periodic shadowing and Ω-stability

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Abstract

We show that the following three properties of a diffeomorphism f of a smooth closed manifold are equivalent: (i) f belongs to the C^1-interior of the set of diffeomorphisms having periodic shadowing property; (ii) f has Lipschitz periodic shadowing property; (iii) f is Ω-stable. Bibliography: 20 titles.

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1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well developed part of the global theory of dynamical systems (see, for example, the monographs [1, 2]).

This theory is closely related to the classical theory of structural stability. It is well known that a diffeomorphism has shadowing property in a neighborhood of a hyberbolic set [3, 4] and a structurally stable diffeomorpism has shadowing property on the whole manifold [5 – 7]. Analyzing the proofs of the first shadowing results by Anosov [3] and Bowen [4], it is easy to see that, in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz (and the same holds in the case of a structurally stable diffeomorphism, see [1]).

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The shadowing property means that, near a sufficiently precise approximate trajectory of a dynamical system, there is an exact trajectory. One can pose a similar question replacing arbitrary approximate and exact trajectories by periodic ones (the corresponding property is called periodic shadowing property, see [8]).

In this paper, we study relations between periodic shadowing and structural stability (to be more precise, Ω-stability).

It is easy to give an example of a diffeomorphism that is not structurally stable but has shadowing property (see [9], for example). Similarly, there exist diffeomorphisms that are not Ω-stable but have periodic shadowing property.

Thus, structural stability is not equivalent to shadowing (and Ω-stability is not equivalent to periodic shadowing).

One of possible approaches in the study of relations between shadowing and structural stability is the passage to C^1-interiors. At present, it is known that the C^1-interior of the set of diffeomorphisms having shadowing property coincides with the set of structurally stable diffeomorphisms [10]. Later, a similar result was obtained for orbital shadowing property (see [11] for details).

In this paper, we show that the C^1-interior of the set of diffeomorphisms having periodic shadowing property coincides with the set of Ω-stable diffeomorphisms.

We are also interested in the study of the above-mentioned relations without the passage to C^1-interiors. Let us mention in this context that Abdenur and Diaz conjectured that a C^1-generic diffeomorphism with shadowing property is structurally stable; they have proved this conjecture for so-called tame diffeomorphisms [12]. Recently, it was proved that Lipschitz shadowing and the so-called variational shadowing are equivalent to structural stability [13, 9].

The second main result of this paper states that Lipschitz periodic shadowing property is equivalent to Ω-stability.

2 Main results

Let us pass to exact definitions and statements.

Let f be a diffeomorphism of a smooth closed manifold M with Riemannian metric dist. We denote by $Df(x)$ the differential of f at a point

$x \in M$.

Denote by T_xM the tangent space of M at a point x; let $|v|$, $v \in T_xM$, be the norm generated by the metric dist.

As usual, we say that a sequence $\xi = \{x_i \in M, \ i \in \mathbb{Z}\}$ is a d-pseudotrajectory of f if

$$\operatorname{dist}(f(x_i), x_{i+1}) < d, \quad i \in \mathbb{Z}. \tag{1}$$

Definition 1. We say that f has *periodic shadowing* property if for any positive ε there exists a positive d such that if $\xi = \{x_i\}$ is a periodic d-pseudotrajectory, then there exists a periodic point p such that

$$\operatorname{dist}(f^i(p), x_i) < \varepsilon, \quad i \in \mathbb{Z}. \tag{2}$$

Denote by PerSh the set of diffeomorphisms having periodic shadowing property.

Definition 2. We say that f has *Lipschitz periodic shadowing* property if there exist positive constants $\mathcal{L}, d_0$ such that if $\xi = \{x_i\}$ is a periodic d-pseudotrajectory with $d \le d_0$, then there exists a periodic point p such that

$$\operatorname{dist}(f^i(p), x_i) \le \mathcal{L}d, \quad i \in \mathbb{Z}. \tag{3}$$

Denote by LipPerSh the set of diffeomorphisms having Lipschitz periodic shadowing property.

Denote by ΩS the set of Ω-stable diffeomorphisms (it is well known that $f \in \Omega S$ if and only if f satisfies Axiom A and the no cycle condition, see, for example, [14]). Denote by $\mathrm{Diff}^1(M)$ the space of diffeomorphisms of M with the C^1 topology. For a set $P \subset \mathrm{Diff}^1(M)$ we denote by $\mathrm{Int}^1(P)$ its C^1-interior.

Let us state our main result.

Theorem. $\mathrm{Int}^1(\mathrm{PerSh}) = \mathrm{LipPerSh} = \Omega S$.

The structure of the paper is as follows. In Sec. 3, we prove the inclusion $\Omega S \subset \mathrm{LipPerSh}$. Of course, this inclusion implies that $\Omega S \subset \mathrm{PerSh}$. Since the set ΩS is C^1-open, we conclude that $\Omega S \subset \mathrm{Int}^1(\mathrm{PerSh})$. In Sec. 4, we prove the inclusion $\mathrm{Int}^1(\mathrm{PerSh}) \subset \Omega S$. In Sec. 5, we prove the inclusion $\mathrm{LipPerSh} \subset \Omega S$.

3 $\Omega S \subset$ LipPerSh

First we introduce some basic notation. Denote by $\mathrm{Per}(f)$ the set of periodic points of f and by $\Omega(f)$ the nonwandering set of f. Let $N = \sup_{x \in M} \|Df(x)\|$.

Let us formulate several auxiliary definitions and statements.

It is well known that if a diffeomorphism f satisfies Axiom A, then its nonwandering set can be represented as a disjoint union of a finite number of compact sets:

$$\Omega(f) = \Omega_1 \cup \dots \cup \Omega_m, \tag{4}$$

where the sets Ω_i are so-called basic sets (hyperbolic sets each of which contains a dense positive semi-trajectory).

We say that a diffeomorphism f has Lipschitz shadowing property on a set U if there exist positive constants $\mathcal{L}, d_0$ such that if $\xi = \{x_i,\ i \in \mathbb{Z}\} \subset U$ is a d-pseudotrajectory with $d \le d_0$, then there exists a point $p \in U$ such that inequalities (3) hold.

We say that a diffeomorphism f is expansive on a set U if there exists a positive number a (expansivity constant) such that if two trajectories $\{f^i(p) : i \in \mathbb{Z}\}$ and $\{f^i(q) : i \in \mathbb{Z}\}$ belong to U and the inequalities

$$\mathrm{dist}(f^i(p), f^i(q)) \le a, \quad i \in \mathbb{Z},$$

hold, then $p = q$.

The following statement is well known (see [1, 14], for example).

Proposition. *If Λ is a hyperbolic set, then there exists a neighborhood U of Λ such that f has Lipschitz shadowing property on U and is expansive on U.*

We also need the following two lemmas (see [15]).

Lemma 1. *Let f be a homeomorpism of a compact metric space (X, dist). For any neighborhood U of the nonwandering set $\Omega(f)$ there exist positive numbers B, d_1 such that if $\xi = \{x_i,\ i \in \mathbb{Z}\}$ is a d-pseudotrajectory of f with $d \le d_1$ and*

$$x_k, x_{k+1}, \dots, x_{k+l} \notin U$$

for some $l > 0$ and $k \in \mathbb{Z}$, then $l \le B$.

Let $\Omega_1, \dots, \Omega_m$ be the basic sets in decomposition (4) of the nonwandering set of an Ω-stable diffeomorphism f.

Lemma 2. *Let $U_1, \dots, U_m$ be disjoint neighborhoods of the basic sets $\Omega_1, \dots, \Omega_m$. There exist neighborhoods $V_j \subset U_j$ of the sets Ω_j and a number $d_2 > 0$ such that if $\xi = \{x_i,\ i \in \mathbb{Z}\}$ is a d-pseudotrajectory of f with $d \le d_2$ such that $x_0 \in V_j$ and $x_t \notin U_j$ for some $j \in \{1, \dots, m\}$ and some $t > 0$, then $x_i \notin V_j$ for $i \ge t$.*

Lemma 3. $\Omega S \subset \mathrm{LipPerSh}$.

Proof. Apply the above proposition and find disjoint neighborhoods $W_1, \dots, W_m$ of the basic sets $\Omega_1, \dots, \Omega_m$ in decomposition (4) such that (i) f has Lipschitz shadowing property on any of W_j with the same constants $\mathcal{L}, d_0^*$; (ii) f is expansive on any of W_j with the same expansivity constant a.

Find neighborhoods V_j, U_j of Ω_j (and reduce d_0^*, if necessary) so that the following properties are fulfilled:

- $V_j \subset U_j \subset W_j, \quad j = 1, \dots, m$;
- the statement of Lemma 2 holds for V_j and U_j with some $d_2 > 0$;
- the $\mathcal{L}d_0^*$-neighborhoods of U_j belong to W_j.

Apply Lemma 1 to find the corresponding constants B, d_1 for the neighborhood $V_1 \cup \dots \cup V_m$ of $\Omega(f)$.

We claim that f has the Lipschitz periodic shadowing property with constants $\mathcal{L}, d_0$, where
$$d_0 = \min\left(d_0^*, d_1, d_2, \frac{a}{2\mathcal{L}}\right).$$

Take a μ-periodic d-pseudotrajectory $\xi = \{x_i,\ i \in \mathbb{Z}\}$ of f with $d \le d_0$. Lemma 1 implies that there exists a neighborhood V_j such that $\xi \cap V_j \neq \emptyset$; shifting indices, we may assume that $x_0 \in V_j$.

In this case, $\xi \subset U_j$. Indeed, if $x_{i_0} \notin U_j$ for some i_0, then $x_{i_0+k\mu} \notin U_j$ for all k. It follows from Lemma 2 that if $i_0 + k\mu > 0$, then $x_i \notin V_j$ for $i \ge i_0 + k\mu$, and we get a contradiction with the periodicity of ξ and the inclusion $x_0 \in V_j$.

Thus, there exists a point p such that inequalities (3) hold. Let us show that $p \in \mathrm{Per}(f)$. By the choice of U_j and W_j, $f^i(p) \in W_j$ for all $i \in \mathbb{Z}$. Let $q = f^\mu(p)$. Inequalities (3) and the periodicity of ξ imply that
$$\mathrm{dist}(f^i(q), x_i) = \mathrm{dist}(f^i(q), x_{i+\mu}) \le \mathcal{L}d, \quad i \in \mathbb{Z}.$$

Thus,
$$\mathrm{dist}(f^i(q), f^i(p)) \le 2\mathcal{L}d \le a, \quad i \in \mathbb{Z},$$
which implies that $f^\mu(p) = q = p$. This completes the proof.

Remark. Thus, we have shown that an Ω-stable diffeomorphism has periodic shadowing property (and its Lipschitz variant). It must be noted that it was shown in [16] that there exist Ω-stable diffeomorphisms that do not have weak shadowing property (hence, they do not have orbital and usual shadowing properties, see [11] for details).

4 $\mathbf{Int}^1(\mathbf{PerSh}) \subset \Omega S$

In the proof, we refer to the following well-known statement. Denote by HP the set of diffeomorphisms f such that every periodic point of f is hyperbolic; let $\mathcal{F} = \mathrm{Int}^1(\mathrm{HP})$. It is known (see [17, 18]) that the set $\mathcal{F}$ coincides with the set ΩS of Ω-stable diffeomorphisms.

Thus, it suffices for us to prove the following statement.

Lemma 4. $\mathrm{Int}^1(\mathrm{PerSh}) \subset \mathcal{F}$.

Proof. In the proof of this lemma, as well as in some proofs below, we apply the usual linearization technique based on exponential mapping.

Let exp be the standard exponential mapping on the tangent bundle of M and let $\exp_x$ be the corresponding mapping

$$T_xM \to M.$$

Let p be a periodic point of f; denote $p_i = f^i(p)$ and $A_i = Df(p_i)$.

We introduce the mappings

$$F_i = \exp_{p_{i+1}}^{-1} \circ f \circ \exp_{p_i} : T_{p_i}M \to T_{p_{i+1}}M. \tag{5}$$

It follows from the standard properties of the exponential mapping that $D\exp_x(0) = \mathrm{Id}$; hence,

$$DF_i(0) = A_i.$$

We can represent

$$F_i(v) = A_i v + \phi_i(v),$$

where

$$\frac{|\phi_i(v)|}{|v|} \to 0 \text{ as } |v| \to 0.$$

Denote by $B(r, x)$ the ball in M of radius r centered at a point x and by $B_T(r, x)$ the ball in T_xM of radius r centered at the origin.

There exists $r > 0$ such that, for any $x \in M$, $\exp_x$ is a diffeomorphism of $B_T(r, x)$ onto its image, and $\exp_x^{-1}$ is a diffeomorphism of $B(r, x)$ onto its image. In addition, we may assume that r has the following property.

If $v, w \in B_T(r, x)$, then

$$\frac{\operatorname{dist}(\exp_x(v), \exp_x(w))}{|v - w|} \le 2;$$

if $y, z \in B(r, x)$, then

$$\frac{|\exp_x^{-1}(y) - \exp_x^{-1}(z)|}{\operatorname{dist}(y, z)} \le 2.$$

Every time, constructing periodic d-pseudotrajectories of f, we take d so small that the considered points of our pseudotrajectories, points of shadowing trajectories, their "lifts" to tangent spaces, etc belong to the corresponding balls $B(r, p_i)$ and $B_T(r, p_i)$ (and we do not repeat this condition on the smallness of d).

To prove Lemma 4, it is enough for us to show that $\operatorname{Int}^1(\mathrm{PerSh}) \subset \mathrm{HP}$ and to note that the left-hand side of this inclusion is C^1-open.

To get a contradiction, let us assume that a diffeomorphism $f \in \operatorname{Int}^1(\mathrm{PerSh})$ has a nonhyperbolic periodic point p. Fix a C^1-neighborhood $\mathcal{N} \subset \mathrm{PerSh}$ of f.

For simplicity, let us assume that p is a fixed point and that the matrix $A_0 = Df(p)$ has an eigenvalue $\lambda = 1$ (the remaining cases are considered using a similar reasoning, see, for example, [19]).

In our case, an analog of mapping (5),

$$F = \exp_p^{-1} \circ f \circ \exp_p : T_pM \to T_pM,$$

has the form

$$F(v) = A_0 v + \phi(v).$$

Clearly, we can find a number $a \in (0, r)$ (recall that the number r was fixed above when properties of the exponential mapping were described), coordinates $v = (u, w)$ in T_pM with one-dimensional u, and a diffeomorphism $h \in \mathcal{N}$ such that if

$$H = \exp_p^{-1} \circ h \circ \exp_p$$

and $|v| \le a$, then

$$H(v) = Av = (u, Bw),$$

where B is a matrix of size $(n-1)\times(n-1)$ (and n is the dimension of M). For this purpose, we take a matrix A, close to A_0 and having an eigenvalue $\lambda = 1$ of multiplicity one, and "annihilate" the C^1-small term $(A_0 - A)v + \phi(v)$ in the small ball $B_T(a, p)$.

Take a positive ε such that $8\varepsilon < a$. Since $h \in \mathcal{N}$, there exists a corresponding $d \in (0, \varepsilon)$ from the definition of periodic shadowing (for the diffeomorphism h). Take a natural number K such that $Kd > 8\varepsilon$. Reducing d, if necessary, we may assume that

$$8\varepsilon < Kd < 2a. \tag{6}$$

Let us construct a sequence $y_k \in T_pM,\ k \in \mathbb{Z}$, as follows:

$$y_0 = 0, \quad y_{k+1} = Ay_k + \left(\frac{d}{2}, 0\right), \quad 0 \le k \le K-1,$$

$$y_{k+1} = Ay_k - \left(\frac{d}{2}, 0\right), \quad K \le k \le 2K-1,$$

and $y_{k+2K} = y_k,\ k \in \mathbb{Z}$. Clearly,

$$y_K = \left(\frac{Kd}{2}, 0\right). \tag{7}$$

Let

$$x_k = \exp_p(y_k).$$

Since

$$\exp_p^{-1}(h(x_k)) = H(y_k) = Ay_k$$

and

$$|y_{k+1} - Ay_k| = \frac{d}{2},$$

the sequence $\xi = \{x_k\}$ is a $2K$-periodic d-pseudotrajectory of h.

By our assumption, there exists a periodic point p_0 of h such that

$$\mathrm{dist}(p_k, x_k) < \varepsilon, \quad k \in \mathbb{Z},$$

where $p_k = h^k(p_0)$. Let

$$p_k = \exp_p(q_k), \quad k \in \mathbb{Z},$$

where $q_k = (U_k, W_k)$, and let $y_k = (u_k, w_k)$; then

$$|U_k - u_k| \le |q_k - y_k| < 2\varepsilon, \quad k \in \mathbb{Z},$$

which implies that

$$|U_0| \le |q_0| < 2\varepsilon.$$

Since $q_{k+1} = H(q_k)$, $U_k = U_0$ for all k due to the structure of H. We conclude that $|U_K| < 2\varepsilon$ and get a contradiction with the inequalities $|U_K - u_K| < 2\varepsilon$, (6), and (7). The lemma is proved.

5 LipPerSh $\subset \Omega S$

In this section, we assume that $f \in \mathrm{LipPerSh}$ (with constants $\mathcal{L} \ge 1, d_0 > 0$). Clearly, in this case $f^{-1} \in \mathrm{LipPerSh}$ as well (and we assume that the constants $\mathcal{L}, d_0$ are the same for f and f^{-1}).

In the construction of pseudotrajectories, we apply the same linearization technique as in the previous section.

Lemma 5. *Every point $p \in \mathrm{Per}(f)$ is hyperbolic.*

Proof. To get a contradiction, let us assume that f has a nonhyperbolic periodic point p (to simplify notation, we assume that p is a fixed point; literally the same reasoning can be applied to a periodic point of period $m > 1$).

In this case, mapping (5) takes the form

$$F(v) = \exp_p^{-1} \circ f \circ \exp_p(v) = Av + \phi(v),$$

where A is a nonhyperbolic matrix. The following two cases are possible:

(Case 1): A has a real eigenvalue λ with $|\lambda| = 1$;

(Case 2): A has a complex eigenvalue λ with $|\lambda| = 1$.

We treat in detail only Case 1; we give a comment concerning Case 2. To simplify presentation, we assume that 1 is an eigenvalue of A; the case of eigenvalue -1 is treated similarly.

We can find coordinates v in T_pM such that, with respect to this coordinate, the matrix A has block-diagonal form,

$$A = \mathrm{diag}(B, P), \tag{8}$$

where B is a Jordan block of size $l \times l$:

$$B = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Of course, introducing new coordinates, we have to change the constants $\mathcal{L}, d_0, N$; we denote the new constants by the same symbols. In addition, we assume that $\mathcal{L}$ is integer.

We start considering the case $l = 2$; in this case,

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$e_1 = (1, 0, 0, \dots, 0) \text{ and } e_2 = (0, 1, 0, \dots, 0)$$

be the first two vectors of the standard orthonormal basis.

Let $K = 25\mathcal{L}$.

Take a small $d > 0$ and construct a finite sequence $y_0, \dots, y_Q$ in T_pM (where Q is determined later) as follows: $y_0 = 0$ and

$$y_{k+1} = Ay_k + de_2, \quad k = 0, \dots, K-1. \tag{9}$$

Then

$$y_K = (Z_1(K)d, Kd, 0, \dots, 0),$$

where the natural number $Z_1(K)$ is determined by K (we do not write $Z_1(K)$ explicitly). Now we set

$$y_{k+1} = Ay_k - de_2, \quad k = K, \dots, 2K-1.$$

Then

$$y_{2K} = (Z_2(K)d, 0, 0, \dots, 0),$$

where the natural number $Z_2(K)$ is determined by K as well. Take $Q = 2K + Z_2(K)$; if we set

$$y_{k+1} = Ay_k - de_1, \quad k = 2K, \dots, Q-1,$$

then $y_Q = 0$. Let us note that both numbers Q and

$$Y := \frac{\max_{0 \le k \le Q-1} |y_k|}{d}$$

are determined by K (and hence, by $\mathcal{L}$).

Now we construct a Q-periodic sequence $y_k, k \in \mathbb{Z}$, that coincides with the above sequence for $k = 0, \dots, Q$.

We set $x_k = \exp_p(y_k)$ and claim that if d is small enough, then $\xi = \{x_k\}$ is a $4d$-pseudotrajectory of f (and this pseudotrajectory is Q-periodic by construction).

Indeed, we know that $|y_k| \le Yd$ for $k \in \mathbb{Z}$. Since $\phi(v) = o(|v|)$ as $|v| \to 0$,

$$|\phi(y_k)| < d, \quad k \in \mathbb{Z}, \tag{10}$$

if d is small enough.

The definition of $\{y_k\}$ implies that

$$|y_{k+1} - Ay_k| = d, \quad k \in \mathbb{Z}. \tag{11}$$

Note that

$$\exp_p^{-1}(f(x_k)) = F(y_k) = Ay_k + \phi(y_k);$$

thus, it follows from (10) and (11) that

$$|y_{k+1} - \exp_p^{-1}(f(x_k))| \le |y_{k+1} - Ay_k| + |\phi(y_k)| < 2d,$$

which implies that $\xi = \{x_k\}$ is a $4d$-pseudotrajectory of f if d is small enough.

Now we estimate the distances between points of trajectories of the mapping F and its linearization.

Let us take a vector $q_0 \in T_pM$ and assume that the sequence $q_k = F^k(q_0)$ belongs to the ball $|v| \le (Y + 8\mathcal{L})d$ for $0 \le k \le K$. Let $r_k = A^k q_0$ (we impose no conditions on r_k since below we estimate ϕ at points q_k only).

Take a small number $\mu \in (0, 1)$ (to be chosen later) and assume that d is small enough, so that the inequality

$$|\phi(v)| \le \mu |v|$$

holds for $|v| \le (Y + 8\mathcal{L})d$.

Then

$$|q_1| \le |Aq_0| + |\phi(q_0)| \le (N+1)|q_0|, \dots, |q_k| \le |Aq_{k-1}| + |\phi(q_{k-1})| \le (N+1)^k |q_0|$$

for $1 \le k \le K$, and

$$|q_1 - r_1| = |Aq_0 + \phi(q_0) - Aq_0| \le \mu |q_0|,$$

$$|q_2 - r_2| = |Aq_1 + \phi(q_1) - Ar_1| \le N|q_1 - r_1| + \mu|q_1| \le \mu(2N+1)|q_0|,$$

$$|q_3 - r_3| \le N|q_2 - r_2| + \mu|q_2| \le \mu(N(2N+1) + (N+1)^2)|q_0|,$$

and so on.

Thus, there exists a number $\nu = \nu(K, N)$ such that

$$|q_k - r_k| \le \mu\nu|q_0|, \quad 0 \le k \le K.$$

We take $\mu = 1/\nu$, note that $\mu = \mu(K, N)$, and get the inequalities

$$|q_k - r_k| \le |q_0|, \quad 0 \le k \le K, \tag{12}$$

for d small enough.

Since $f \in$ LipPerSh, for d small enough, the Q-periodic $4d$-pseudotrajectory ξ is $4\mathcal{L}d$-shadowed by a periodic trajectory. Let p_0 be a point of this trajectory such that

$$\operatorname{dist}(p_k, x_k) \le 4\mathcal{L}d, \quad k \in \mathbb{Z}, \tag{13}$$

where $p_k = f^k(p_0)$. Let $q_k = \exp_p^{-1}(p_k)$.

The inequalities $|y_k| \le Yd$ and (13) imply that

$$|q_k| \le |y_k| + 2\operatorname{dist}(p_k, x_k) \le (Y + 8\mathcal{L})d, \quad k \in \mathbb{Z}. \tag{14}$$

Note that $|q_0| \le 8\mathcal{L}d$.

Set $r_k = A^k q_0$; we deduce from estimate (12) that if d is small enough, then

$$|q_K - r_K| \le |q_0| \le 8\mathcal{L}d. \tag{15}$$

Denote by $v^{(2)}$ the second coordinate of a vector $v \in T_pM$.

It follows from the structure of the matrix A that

$$|r_K^{(2)}| = |q_0^{(2)}| \le 8\mathcal{L}d. \tag{16}$$

The relations

$$|y_K^{(2)}| = Kd \text{ and } |q_K - y_K| \le 8\mathcal{L}d$$

imply that

$$|q_K^{(2)}| \geq Kd - 8\mathcal{L}d = 17\mathcal{L}d \qquad (17)$$

(recall that $K = 25\mathcal{L}$).

Estimates (15)–(17) are contradictory. Our lemma is proved in Case 1 for $l = 2$.

If $l = 1$, then the proof is simpler; the first coordinate of $A^k v$ equals the first coordinate of v, and we construct the periodic pseudotrajectory perturbing the first coordinate only.

If $l > 2$, the reasoning is parallel to that above; we first perturb the lth coordinate to make it Kd, and then produce a periodic sequence consequently making zero the lth coordinate, the $(l-1)$st coordinate, and so on.

If λ is a complex eigenvalue, $\lambda = a + bi$, we take a real 2×2 matrix

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and assume that in representation (8), B is a real $2l \times 2l$ Jordan block:

$$B = \begin{pmatrix} R & E_2 & 0 & \dots & 0 \\ 0 & R & E_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R \end{pmatrix},$$

where E_2 is the 2×2 unit matrix.

After that, almost the same reasoning works; we note that $|Rv| = |v|$ for any 2-dimensional vector v and construct periodic pseudotrajectories replacing, for example, formulas (9) by the formulas

$$y_{k+1} = Ay_k + dw_k, \quad k = 0, \dots, K-1,$$

where jth coordinates of the vector w_k are zero for $j = 1, \dots, 2l-2, 2l+1, \dots, n$, while the 2-dimensional vector corresponding to $(2l-1)$st and $2l$th coordinates has the form $R^k w$ with $|w| = 1$, and so on. We leave details to the reader. The lemma is proved.

Lemma 6. *There exist constants $C > 0$ and $\lambda \in (0,1)$ depending only on N and $\mathcal{L}$ and such that, for any point $p \in \mathrm{Per}(f)$, there exist complementary subspaces $S(p)$ and $U(p)$ of the tangent space T_pM that are Df-invariant, i.e.,*

(H1) $Df(p)S(p) = S(f(p))$ *and* $Df(p)U(p) = U(f(p))$*,*
and the inequalities
(H2.1) $|Df^j(p)v| \le C\lambda^j|v|, \quad v \in S(p), j \ge 0,$
and
(H2.2) $|Df^{-j}(p)v| \le C\lambda^j|v|, \quad v \in U(p), j \ge 0,$
hold.

Remark. Lemma 6 means that the set $\mathrm{Per}(f)$ has all the standard properties of a hyperbolic set, with the exception of compactness.

Proof. Take a periodic point $p \in \mathrm{Per}(f)$; let m be the minimal period of p.

Denote $p_i = f^i(p)$, $A_i = Df(p_i)$, and $B = Df^m(p)$. It follows from Lemma 5 that the matrix B is hyperbolic. Denote by $S(p)$ and $U(p)$ the invariant subspaces of B corresponding to parts of its spectrum inside and outside the unit disk, respectively. Clearly, $S(p)$ and $U(p)$ are invariant with respect to Df, $T_pM = S(p) \oplus U(p)$, and the following relations hold:

$$\lim_{n\to+\infty} B^n v_s = \lim_{n\to+\infty} B^{-n} v_u = 0, \quad v_s \in S(p), v_u \in U(p). \tag{18}$$

We prove that inequalities (H2.2) hold with $C = 16\mathcal{L}$ and $\lambda = 1 + 1/(8\mathcal{L})$ (inequalities (H2.1) are established by similar reasoning applied to f^{-1} instead of f).

Consider an arbitrary nonzero vector $v_u \in U(p)$ and an integer $j \ge 0$. Define sequences $v_i, e_i \in T_{p_i}M$ and $\lambda_i > 0$ for $i \ge 0$ as follows:

$$v_0 = v_u, \quad v_{i+1} = A_i v_i, \quad e_i = \frac{v_i}{|v_i|}, \quad \lambda_i = \frac{|v_{i+1}|}{|v_i|} = |A_i e_i|.$$

Let

$$\tau = \frac{\lambda_{m-1}\cdot\ldots\cdot\lambda_1 + \lambda_{m-1}\cdot\ldots\cdot\lambda_2 + \ldots + \lambda_{m-1} + 1}{\lambda_{m-1}\cdot\ldots\cdot\lambda_0}.$$

Consider the sequence $\{a_i \in \mathbb{R},\ i \ge 0\}$ defined by the following formulas:

$$a_0 = \tau, \quad a_{i+1} = \lambda_i a_i - 1. \tag{19}$$

Note that

$$a_m = 0 \quad \text{and} \quad a_i > 0, \quad i \in [0, m-1]. \tag{20}$$

Indeed, if $a_i \le 0$ for some $i \in [0, m-1]$, then $a_k < 0$ for $k \in [i+1, m]$.

It follows from (18) that there exists $n > 0$ such that

$$|B^{-n}\tau e_0| < 1. \tag{21}$$

Consider the finite sequence $\{w_i \in T_{p_i}M,\ i \in [0, m(n+1)]\}$ defined as follows:

$$\begin{cases} w_i = a_i e_i, & i \in [0, m-1], \\ w_m = B^{-n}\tau e_0, & \\ w_{m+1+i} = A_i w_{m+i}, & i \in [0, mn-1]. \end{cases}$$

Clearly,

$$w_{km} = B^{k-1-n}\tau e_0, \quad k \in [1, n+1],$$

which means that we can consider $\{w_i\}$ as an $m(n+1)$-periodic sequence defined for $i \in \mathbb{Z}$.

Let us note that

$$A_i w_i = a_i A_i e_i = a_i \frac{v_{i+1}}{|v_i|}, \quad i \in [0, m-2],$$

$$w_{i+1} = (\lambda_i a_i - 1)\frac{v_{i+1}}{|v_{i+1}|} = a_i \frac{v_{i+1}}{|v_i|} - e_{i+1}, \quad i \in [0, m-2],$$

and

$$A_{m-1} w_{m-1} = a_{m-1}\frac{v_m}{|v_{m-1}|} = \frac{v_m}{\lambda_{m-1}|v_{m-1}|} = e_m$$

(in the last relation we take into account that $a_{m-1}\lambda_{m-1} = 1$ since $a_m = 0$).

The above relations and condition (21) imply that

$$|w_{i+1} - A_i w_i| < 2, \quad i \in \mathbb{Z}. \tag{22}$$

Now we take a small $d > 0$ and consider the $m(n+1)$-periodic sequence $\xi = \{x_i = \exp_{p_i}(dw_i),\ i \in \mathbb{Z}\}$.

We claim that if d is small enough, then ξ is a $4d$-pseudotrajectory of f.

Denote

$$\zeta_{i+1} = \exp_{p_{i+1}}^{-1}(f(x_i)) \ \text{ and } \ \zeta'_{i+1} = \exp_{p_{i+1}}^{-1}(x_{i+1}).$$

Then

$$\zeta_{i+1} = \exp_{p_{i+1}}^{-1} f(\exp_{p_i}(dw_i)) = F_i(dw_i) = A_i dw_i + \phi_i(dw_i),$$

where the mapping F_i is defined in (5) and $\phi_i(v) = o(|v|)$, and

$$\zeta'_{i+1} = \exp^{-1}_{p_{i+1}}(x_{i+1}) = dw_{i+1}.$$

It follows from estimates (22) that

$$|\zeta'_{i+1} - \zeta_{i+1}| \leq 2d$$

for small d, and

$$\mathrm{dist}(f(x_i), x_{i+1}) \leq 4d.$$

By Lemma 5, the m-periodic trajectory $\{p_i\}$ is hyperbolic; hence, $\{p_i\}$ has a neighborhood in which $\{p_i\}$ is a unique periodic trajectory. It follows that if d is small enough, then the pseudotrajectory $\{x_i\}$ is $4\mathcal{L}d$-shadowed by $\{p_i\}$.

The inequalities $\mathrm{dist}(x_i, p_i) \leq 4\mathcal{L}d$ imply that $|a_i| = |w_i| \leq 8\mathcal{L}$ for $0 \leq i \leq m-1$.

Now the equalities $\lambda_i = (a_{i+1}+1)/a_i$ imply that if $0 \leq i \leq m-1$, then

$$\lambda_0 \cdot \ldots \cdot \lambda_{i-1} = \frac{a_1+1}{a_0}\frac{a_2+1}{a_1}\ldots\frac{a_i+1}{a_{i-1}} =$$

$$= \frac{a_i+1}{a_0}\left(1+\frac{1}{a_1}\right)\ldots\left(1+\frac{1}{a_{i-1}}\right) \geq$$

$$\geq \frac{1}{8\mathcal{L}}\left(1+\frac{1}{8\mathcal{L}}\right)^{i-1} > \frac{1}{16\mathcal{L}}\left(1+\frac{1}{8\mathcal{L}}\right)^{i}$$

(we take into account that $1 + 1/(8\mathcal{L}) < 2$ since $\mathcal{L} \geq 1$).

It remains to note that

$$|Df^i(p)v_u| = \lambda_{i-1}\cdots\lambda_0|v_u|, \quad 0 \leq i \leq m-1,$$

and that we started with an arbitrary vector $v_u \in U(p)$.

This proves our statement for $j \leq m-1$. If $j \geq m$, we take an integer $k > 0$ such that $km > j$ and repeat the above reasoning for the periodic trajectory $p_0, \ldots, p_{km-1}$ (note that we have not used the condition that m is the minimal period). Lemma 6 is proved.

Lemma 7. *If $f \in \mathrm{LipPerSh}$, then f satisfies Axiom A.*

Proof. Denote by P_l the set of points $p \in \mathrm{Per}(f)$ of index l (as usual, the index of a hyperbolic periodic point is the dimension of its unstable manifold).

Let R_l be the closure of P_l. Clearly, R_l is a compact f-invariant set. We claim that any R_l is a hyperbolic set. Let $n = \dim M$.

Consider a point $q \in R_l$ and fix a sequence of points $p_m \in P_l$ such that $p_m \to q$ as $m \to \infty$. By Lemma 6, there exist complementary subspaces $S(p_m)$ and $U(p_m)$ of $T_{p_m}M$ (of dimensions $n-l$ and l, respectively) for which estimates (H2.1) and (H2.2) hold.

Standard reasoning shows that, introducing local coordinates in a neighborhood of (q, T_qM) in the tangent bundle of M, we can select a subsequence p_{m_k} for which the sequences $S(p_{m_k})$ and $U(p_{m_k})$ converge (in the Grassmann topology) to subspaces of T_qM (let S_0 and U_0 be the corresponding limit subspaces).

The limit subspaces S_0 and U_0 are complementary in T_qM. Indeed, consider the "angle" β_{m_k} between the subspaces $S(p_{m_k})$ and $U(p_{m_k})$ which is defined (with respect to the introduced local coordinates in a neighborhood of (q, T_qM)) as follows:

$$\beta_{m_k} = \min |v^s - v^u|,$$

where the minimum is taken over all possible pairs of unit vectors $v^s \in S(p_{m_k})$ and $v^u \in U(p_{m_k})$.

It is shown in [16, Lemma 12.1] that the values β_{m_k} are estimated from below by a positive constant $\alpha = \alpha(C, \lambda, N)$. Clearly, this implies that the subspaces S_0 and U_0 are complementary.

It is easy to show that the limit subspaces S_0 and U_0 are unique (which means, of course, that the sequences $S(p_m)$ and $U(p_m)$ converge). For the convenience of the reader, we prove this statement (our reasoning is close to that of [16]).

To get a contradiction, assume that there is a subsequence p_{m_i} for which the sequences $S(p_{m_i})$ and $U(p_{m_i})$ converge to complementary subspaces S_1 and U_1 different from S_0 and U_0 (for definiteness, we assume that $S_0 \setminus S_1 \neq \emptyset$).

Due to the continuity of Df, the inequalities

$$|Df^j(q)v| \leq C\lambda^j |v|, \quad v \in S_0 \cup S_1,$$

and

$$|Df^j(q)v| \geq C^{-1}\lambda^{-j} |v|, \quad v \in U_0 \cup U_1,$$

hold for $j \geq 0$.

Since

$$T_qM = S_0 \oplus U_0 = S_1 \oplus U_1,$$

our assumption implies that there is a vector $v \in S_0$ such that

$$v = v^s + v^u, \quad v^s \in S_1, v^u \in U_1, v^u \neq 0.$$

Then

$$|Df^j(q)v| \leq C\lambda^j |v| \to 0, \quad j \to \infty,$$

and

$$|Df^j(q)v| \geq C^{-1}\lambda^{-j}|v^u| - C\lambda^j|v^s| \to \infty, \quad j \to \infty,$$

and we get the desired contradiction.

It follows that there are uniquely defined complementary subspaces $S(q)$ and $U(q)$ for $q \in R_l$ with proper hyperbolity estimates; the Df-invariance of these subspaces is obvious. We have shown that each R_l is a hyperbolic set with $\dim S(q) = n - l$ and $\dim U(q) = l$ for $q \in R_l$.

If $r \in \Omega(f)$, then there exists a sequence of points $r_m \to r$ as $m \to \infty$ and a sequence of indices $k_m \to \infty$ as $m \to \infty$ such that $f^{k_m}(r_m) \to r$.

Clearly, if we continue the sequence

$$r_m, f(r_m), \dots, f^{k_m - 1}(r_m)$$

periodically with period k_m, we get a periodic d_m-pseudotrajectory of f with $d_m \to 0$ as $m \to \infty$.

Since $f \in \text{LipPerSh}$, for large m there exist periodic points p_m such that $\text{dist}(p_m, r_m) \to 0$ as $m \to \infty$. Thus, periodic points are dense in $\Omega(f)$.

Since hyperbolic sets with different dimensions of the subspaces $U(q)$ are disjoint, we get the equality

$$\Omega(f) = R_0 \cup \cdots \cup R_n,$$

which implies that $\Omega(f)$ is hyperbolic. The lemma is proved.

It was mentioned above that if a diffeomorphism f satisfies Axiom A, then its nonwandering set can be represented as a disjoint union of a finite number of basic sets (see representation (4)).

The basic sets Ω_i have stable and unstable "manifolds":

$$W^s(\Omega_i) = \{x \in M : \ \text{dist}(f^k(x), \Omega_i) \to 0, \quad k \to \infty\}$$

and

$$W^u(\Omega_i) = \{x \in M: \ \mathrm{dist}(f^k(x), \Omega_i) \to 0, \quad k \to -\infty\}.$$

If Ω_i and Ω_j are basic sets, we write $\Omega_i \to \Omega_j$ if the intersection

$$W^u(\Omega_i) \cap W^s(\Omega_j)$$

contains a wandering point.

We say that f has a 1-cycle if there is a basic set Ω_i such that $\Omega_i \to \Omega_i$.

We say that f has a t-cycle if there are $t > 1$ basic sets

$$\Omega_{i_1}, \dots, \Omega_{i_t}$$

such that

$$\Omega_{i_1} \to \dots \to \Omega_{i_t} \to \Omega_{i_1}.$$

Lemma 8. *If* $f \in \mathrm{LipPerSh}$, *then* f *has no cycles.*

Proof. To simplify presentation, we prove that f has no 1-cycles (in the general case, the idea is literally the same, but the notation is heavy).

To get a contradiction, assume that

$$p \in (W^u(\Omega_i) \cap W^s(\Omega_i)) \setminus \Omega(f).$$

In this case, there are sequences of indices $j_m, k_m \to \infty$ as $m \to \infty$ such that

$$f^{-j_m}(p), f^{k_m}(p) \to \Omega_i, \quad m \to \infty.$$

Since the set Ω_i is compact, we may assume that

$$f^{-j_m}(p) \to q \in \Omega_i \ \text{ and } f^{k_m}(p) \to r \in \Omega_i.$$

Since Ω_i contains a dense positive semi-trajectory, there exist points $s_m \to r$ and indices $l_m > 0$ such that $f^{l_m}(s_m) \to q$ as $m \to \infty$.

Clearly, if we continue the sequence

$$p, f(p), \dots, f^{k_m - 1}(p), s_m, \dots, f^{l_m - 1}(s_m), f^{-j_m}(p), \dots, f^{-1}(p)$$

periodically with period $k_m + l_m + j_m$, we get a periodic d_m-pseudotrajectory of f with $d_m \to 0$ as $m \to \infty$.

Since $f \in \mathrm{LipPerSh}$, there exist periodic points p_m (for m large enough) such that $p_m \to p$ as $m \to \infty$, and we get the desired contradiction with the assumption that $p \notin \Omega(f)$. The lemma is proved.

Lemmas 5 – 8 show that $\mathrm{LipPerSh} \subset \Omega S$.

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